A *Fast* Fourier Transform with Rectangular Output on the BCC and FCC Lattices

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SAMPTA'09

Outline

Optimal Trivariate Sampling

Sphere Packing

- Sampling in spatial domain $\longrightarrow$ Spectrum replication in Fourier domain
- Optimal sampling $\longrightarrow$ Tightest sphere packing
- FCC: densest packing $\longrightarrow$ BCC: optimal sampling

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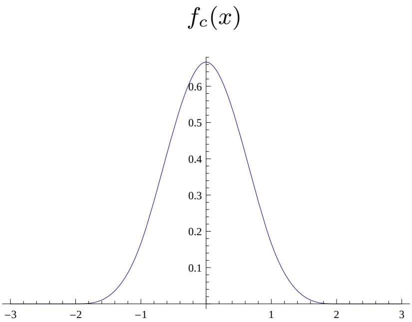
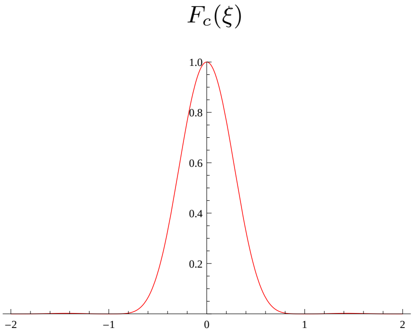
- FCC: densest packing $\longrightarrow$ BCC: optimal sampling

Related Work

- Reconstruction kernels well studied, e.g Box splines: Entezari *et al.* 2008
- Discrete tools still under development
- MDFT for arbitrary lattices, Mersereau *et al.* 1983
- BCC DFT: Csébfalvi *et al. 2008.* Redundant representation

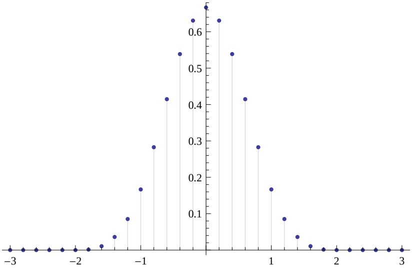
Overview of the DFT



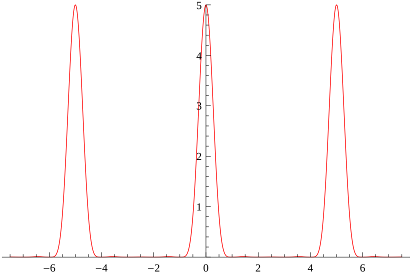


Overview of the DFT

$$f(n) = f_c(Tn)$$

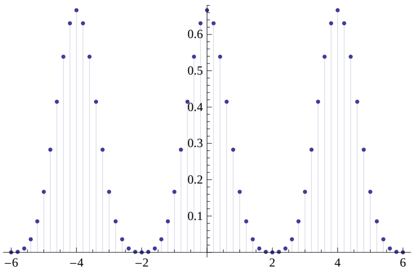


$$\hat{F}(\xi) = \frac{1}{T}\sum_n F_c(\xi - \frac{n}{T})$$

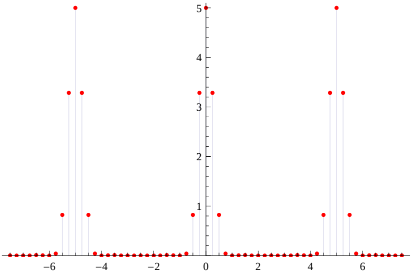


Overview of the DFT

$$f(n+Nk) = f(n)$$

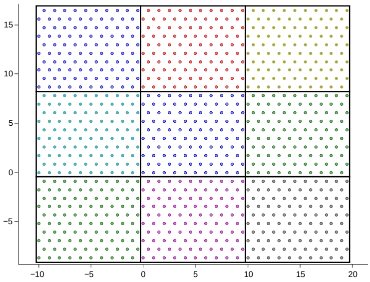


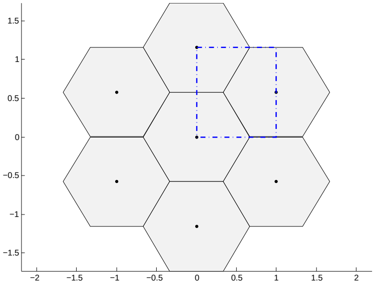
$$F(k) = \hat{F}\left(\frac{k}{NT}\right)$$



Multidimensional DFT

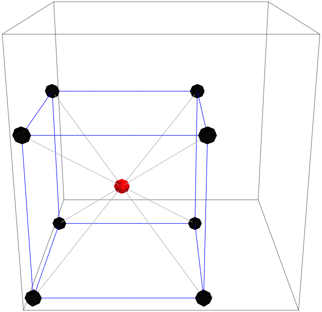
- Multidimensional extension of the DFT
- Sampling: $f(\boldsymbol{n}) = f_c(\boldsymbol{L}\boldsymbol{n})$
  Replication: $\hat{F}(\boldsymbol{\xi}) = \frac{1}{|\det \boldsymbol{L}|} \sum_{\boldsymbol{r}} F_c(\boldsymbol{\xi} - \boldsymbol{L}^{-T}\boldsymbol{r})$
- Periodization of $f(\boldsymbol{n}) \to$ Sampling of $\hat{F}(\boldsymbol{\xi})$



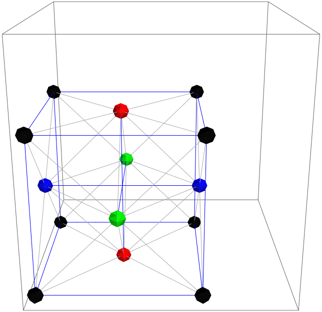


BCC and FCC Sampling

$$\boldsymbol{L}_{\mathsf{B}} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$



$$\boldsymbol{L}_{\mathsf{F}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



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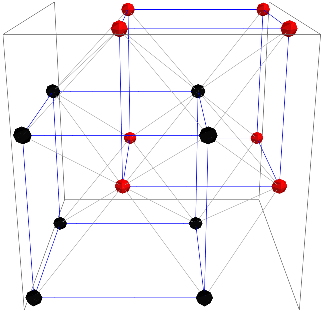

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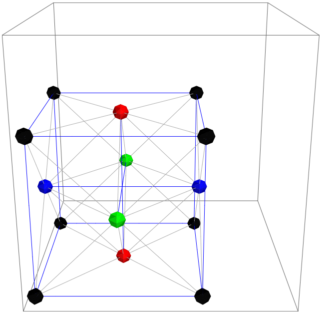

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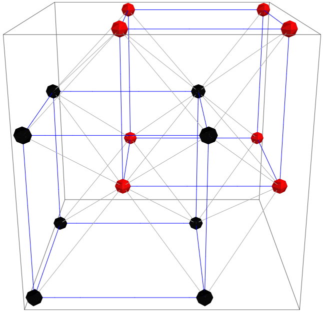

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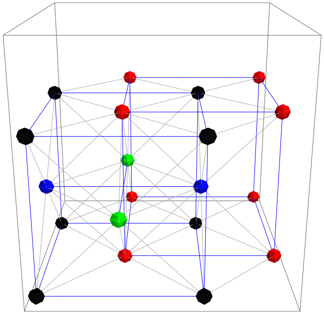

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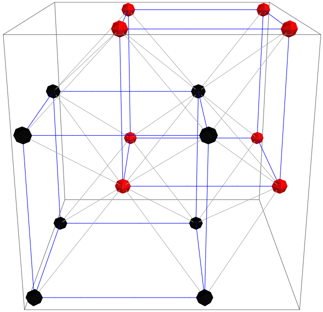

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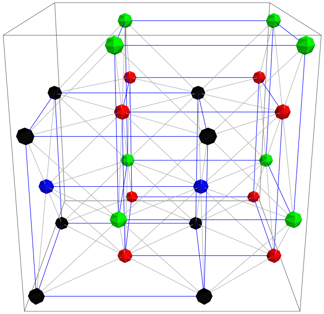

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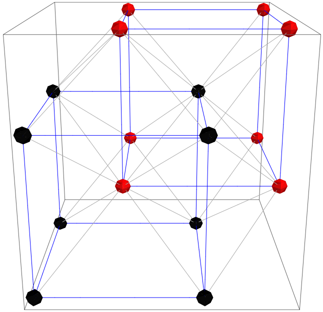

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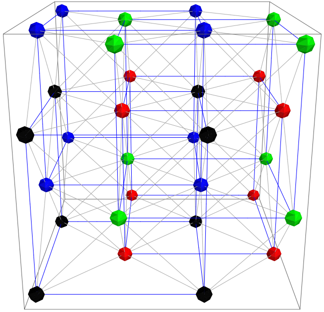


Cartesian Periodicity in Spatial Domain

- Split sampled sequence into two Cartesian sequences:

$$f_0(\boldsymbol{n}) = f_c(2h\boldsymbol{I}\boldsymbol{n}) \quad \text{and} \quad f_1(\boldsymbol{n}) = f_c(2h\boldsymbol{I}\boldsymbol{n} + h\boldsymbol{t})$$

$\boldsymbol{t} = (1, 1, 1)^T$

- Extend both sequences periodically on a Cartesian lattice:

$$f_0(\boldsymbol{n} + \boldsymbol{N}\boldsymbol{r}) = f_0(\boldsymbol{n}) \quad \text{and} \quad f_1(\boldsymbol{n} + \boldsymbol{N}\boldsymbol{r}) = f_1(\boldsymbol{n})$$

$\boldsymbol{N} = \mathrm{diag}(N_1, N_2, N_3)$

- Cuboid fundamenetal region, $N = 2N_1N_2N_3$ samples, volume: $4Nh^3$

Forward BCC Transform

- Cartesian sampling in the Fourier domain:

$$\begin{aligned} F(\boldsymbol{k}) =& \hat{F}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}=\frac{1}{2h}\boldsymbol{N}^{-1}\boldsymbol{k}} \\ =& \sum_{\boldsymbol{n}\in\mathcal{N}} \big(f_0(\boldsymbol{n}) + f_1(\boldsymbol{n})\exp\big[-\pi j \boldsymbol{k}^T\boldsymbol{N}^{-1}\boldsymbol{t}\big]\big) \cdot \exp\big[-2\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1}\boldsymbol{n}\big] \end{aligned}$$

- Transform periodic on FCC lattice:

$$\hat{F}\big(\frac{1}{2h}(\boldsymbol{N}^{-1}\boldsymbol{k} + \boldsymbol{L}_{\mathsf{F}}\boldsymbol{r})\big) = \hat{F}(\frac{1}{2h}\boldsymbol{N}^{-1}\boldsymbol{k})$$

- $F(\boldsymbol{k})$ also Cartesian periodic: $F(\boldsymbol{k} + 2\boldsymbol{N}\boldsymbol{r}) = F(\boldsymbol{k})$

Efficient Evaluation

$$F(\boldsymbol{k}) = \sum_{\boldsymbol{n} \in \mathcal{N}} \left(f_0(\boldsymbol{n}) + f_1(\boldsymbol{n}) \exp\left[-\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1} \boldsymbol{t}\right] \right) \cdot \exp\left[-2\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1} \boldsymbol{n}\right]$$

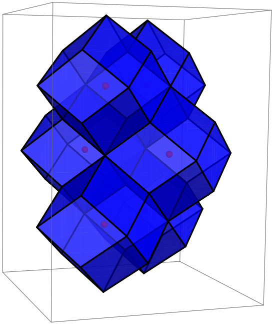
- $\boldsymbol{N}$ is diagonal, kernel therefore separable
- Allows the application of the tensor product FFT

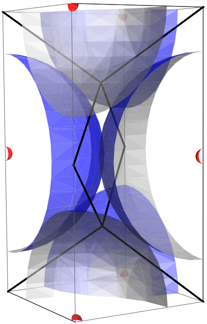
How to choose evaluation region?

- Either exploit Cartesian periodicity in the Fourier domain, $4N$ samples, four fold redundancy
- Or exploit FCC geometry and choose non-redundant Cartesian region that contains the full rhombic dodecahedral period, more efficient

Non-redundant Evaluation

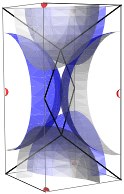
- Six lattice sites contribute to non-redundant region





- Each rhombic dodecahedron contains N samples.

Forward BCC Transform



Split $F(\boldsymbol{k})$ into $F_0(\boldsymbol{k})$ and $F_1(\boldsymbol{k})$

$$F_i(\boldsymbol{k}) := \sum_{\boldsymbol{n}\in\mathcal{N}} f_0(\boldsymbol{n}) \exp\left[-2\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1}\boldsymbol{n}\right] +$$
$$(-1)^i \exp[-\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1}\boldsymbol{t}] \sum_{\boldsymbol{n}\in\mathcal{N}} f_1(\boldsymbol{n}) \exp\left[-2\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1}\boldsymbol{n}\right]$$

- Requires the FFT of the Cartesian sequences $f_0(\boldsymbol{n})$ and $f_1(\boldsymbol{n})$

Inverse BCC Transform

- Likewise, need two inverse FFT evaluations

$$f_0(\boldsymbol{n}) = \frac{1}{N}\sum_{\boldsymbol{k}\in\mathcal{N}}(F_0(\boldsymbol{k}) + F_1(\boldsymbol{k})) \qquad \cdot\exp\big[2\pi j\boldsymbol{k}^T\boldsymbol{N}^{-1}\boldsymbol{n}\big]$$

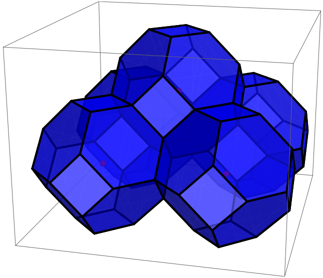
$$f_1(\boldsymbol{n}) = \frac{1}{N}\sum_{\boldsymbol{k}\in\mathcal{N}}\Big((F_0(\boldsymbol{k}) - F_1(\boldsymbol{k}))\exp[\pi j\boldsymbol{k}^T\boldsymbol{N}^{-1}\boldsymbol{t}]\Big) \qquad \cdot\exp\big[2\pi j\boldsymbol{k}^T\boldsymbol{N}^{-1}\boldsymbol{n}\big]$$

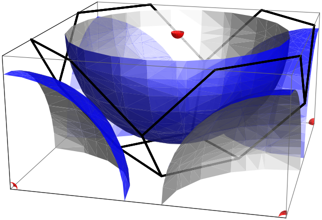
FCC Transform

- Similar derivation
- Split sampled sequence into four Cartesian sequences
- Transform is periodic on a BCC lattice as well as a Cartesian lattice with a two-fold redundancy
- Fundamental period contained within a truncated octahedron
- To eliminate redundancy, choose a suitable Cartesian region that contains one complete truncated octahedral period

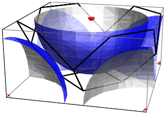
Non-redundant Region

- Five lattice sites contribute to non-redundant region





Evaluation



Split non-redundant region into four Cartesian subregions

Forward

$$F_m(\boldsymbol{k}) = \sum_{i=0}^{3} H_{im} \exp[-\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1} \boldsymbol{t}_i] \Big(\sum_{\boldsymbol{n} \in \mathcal{N}} f_i(\boldsymbol{n}) \exp\big[-2\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1} \boldsymbol{n}\big] \Big)$$

Inverse

$$f_i(\boldsymbol{n}) = \frac{1}{N} \sum_{\boldsymbol{k} \in \mathcal{N}} \Big(\exp[\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1} \boldsymbol{t}_i] \sum_{m=0}^{3} H_{im} F_m(\boldsymbol{k}) \Big) \exp\big[2\pi j \boldsymbol{k}^T \boldsymbol{N}^{-1} \boldsymbol{n}\big]$$

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

- Four FFTs for forward and four IFFTs for inverse

Summary

- Data usually acquired in axis-aligned windows, Cartesian periodicity is therefore practical
- Cartesian periodicity in spatial domain → Cartesian sampling in Fourier domain
- Separable transforms, choose a suitable rectangular region
- Cubic region in Fourier domain redundant
- Non-redundant rectangular regions lead to efficient transforms

Acknowledgements

1. National Science and Engineering Research Council of Canada
2. SAMPTA09 Organizing Committee

Thank you for your attention

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