Rendering in Shift-Invariant Spaces

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Motivation

• Image function describes a continuous 2D image.

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R} \]
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- In rendering, we typically use *area sampling* to discretize the image function \( f \).
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• Image function describes a continuous 2D image.
  \[ f : \mathbb{R}^2 \rightarrow \mathbb{R} \]

• In rendering, we typically use area sampling to discretize the image function \( f \).

• This yields a discrete approximation of \( f \).
  (can be regarded as continuous in light of Shannon’s sampling theory)
Goals

• Use recent developments in signal processing and approximation theory to seek higher-quality grid-based approximations.

• Approximate $f$ in a shift-invariant space.

• Seek an approximation $f_{\text{app}}$ that minimizes the $L_2$ error:
  \[ \|f - f_{\text{app}}\|_{L_2(\mathbb{R}^2)}. \]

• Investigate the quality improvements in the context of ray-tracing.
Outline

• Motivation

• **Shift-Invariant (SI) spaces**

• Rendering in spaces generated by the uniform B-splines

• Results

• Conclusion
Use in Graphics

• More familiar to the image-processing community [Unser 2000]

• Some recent applications in scientific visualization:
  - volume reconstruction [Entezari et al. 2008]
  - gradient estimation [Alim et al. 2010]

• In rendering, antialiasing approaches are largely based on Shannon’s sampling theory. SI spaces offer a much more flexible alternative.
Shift-invariant spaces (1D)

- A vector-space spanned by the (scaled) shifts of a generating function:

\[ \mathbb{V}(\varphi) := \left\{ g(x) = \sum_{n \in \mathbb{Z}} c[n] \varphi(x - n) : c \in l_2(\mathbb{Z}) \right\} \]

- The sequence \( \{\varphi(x - n)\}_{n \in \mathbb{Z}} \) forms a basis for \( \mathbb{V}(\varphi) \).
  (The basis may not be orthogonal)

- There exists a dual basis \( \{\hat{\varphi}(x - n)\}_{n \in \mathbb{Z}} \) that also spans \( \mathbb{V}(\varphi) \). Primal and dual generators are bi-orthonormal:

\[ \langle \varphi, \hat{\varphi}(\cdot - n) \rangle = \delta[n] \]
Familiar Examples

- Shannon’s sampling theory is a special case.

\[ \varphi(x) = \text{sinc}(x) \]

- Interpolating
- Infinite support (not practical)
- Self-dual
- Can represent all bandlimited functions
Familiar Examples

• Shannon’s sampling theory is a special case.

\[ \varphi(x) = \text{sinc}(x) \]

- interpolating
- infinite support (not practical)
- self-dual
- can represent all bandlimited functions

• Cubic interpolation [Mitchell Netravali 88]

\[ \varphi(x) \]

- interpolating
- compact support (practical)
- primal and dual are piecewise cubic
- can represent all cubic polynomials
Minimum-error Approximation

- Minimum-error approximation is obtained by orthogonally projecting $f$ to the desired space $\mathcal{V}(\varphi)$.

- Image approximation:
  \[
  f_{\text{app}}(x) = \sum_n c[n] \varphi(x - n).
  \]

- Measure the coefficients according to:
  \[
  c[n] = \langle f, \varphi(\cdot - n) \rangle
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(\varphi does not need to be an interpolating generator)
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- Image approximation:
  
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  (the dual $\varphi$ is usually not compactly supported)
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Uniform Centered B-splines

- The uniform centered B-spline $b_d(x)$ (degree $d$) is obtained via $d$ convolutions of the box function.

\[ b_d(x) : = (b * b * \ldots * b)(x) \]
\[ = (b * b_{d-1})(x) \]

- For example, the cubic B-spline can be obtained in two different ways:
Uniform Centered B-splines

- The uniform centered B-spline $b_d(x)$ ($d$-degree) is obtained via $d$ convolutions of the box function.

\[
b_d(x) := (b * b * \ldots * b)(x)
\]

\[
d + 1 \text{ repetitions of } b
\]

\[
= (b * b_{d-1})(x)
\]

- For example, the cubic B-spline can be obtained in two different ways:
Dual Generators

• The box function is self-dual. Higher degree B-splines are not.

• Duals have the following Fourier-domain expression:

$$\hat{\varphi}(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{a}_\varphi(\omega)}$$
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DTFT of the autocorrelation sequence. Can be computed analytically for the B-splines.
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• In the spatial domain:

\[ \varphi(x) = \sum_n a_{\varphi}^{-1}[n] \varphi(x - n) \]
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  \[
  {\varphi}(x) = \sum_n a_\varphi^{-1}[n] \varphi(x - n)
  \]
Rendering

- Recall that we wish to compute:

\[ c[n] = \langle f, \varphi(\cdot - n) \rangle \]

- Replacing the dual with its primal representation, we get:

\[
\begin{align*}
    c[n] &= \sum_{m} a_{\varphi}^{-1}[m] \langle f, \varphi(\cdot - (n - m)) \rangle \\
    &= (r * a_{\varphi}^{-1})[n]
\end{align*}
\]
Rendering

• Recall that we wish to compute:

$$c[n] = \langle f, \varphi(\cdot - n) \rangle$$

• Replacing the dual with its primal representation, we get:

$$c[n] = \sum_m a^{-1}_\varphi[m] \langle f, \varphi(\cdot - (n - m)) \rangle$$

$$= (\mathcal{R} * a^{-1}_\varphi)[n]$$

analog acquisition digital processing
The case since we can use the primal representation of the dual product in Eq. (4) is an expensive operation. However, this is not supported, which might lead one to believe that the evaluation of the inner product (cf. Eq. (2)) is a leverage. In particular, we have designed to counter the effect of the non-ideal acquisition [3].

As before, for the sake of clarity, we focus on the univariate sequences. Additionally, without loss of generality, we assume that the acquired sequence

where $\phi_m[n]$ is the sequence obtained by sampling the generator $\phi$ when $\phi$ is not compactly supported, the sequence

is compactly supported, the sequence

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where

Figure 2 summarizes this two-stage rendering pipeline. By retaining the box filter that is widely used in practice does indeed perform the minimum-error box, tent and cubic filters. Observe that

that can be readily incorporated in an existing renderer with very little processing step. However, the similarity ends there since the remaining spaces require a non-trivial digital processing step.

4.1 Example spaces

Table 1: Example spaces used in our experiments

rendering in B-spline spaces
The rendering process therefore consists of an analog acquisition step followed by a digital processing step. Once the approximation coefficients have been obtained, the digital rendering operation can freely choose the PSF (anti-aliasing prefilter). In this section, we assume that the PSF can be designed to counter the effect of the non-ideal acquisition.

Table 1: Example spaces used in our experiments

<table>
<thead>
<tr>
<th>Name</th>
<th>Polynomial pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>0</td>
</tr>
<tr>
<td>Cubic</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>Quadratic</td>
<td>0, 1, 2</td>
</tr>
<tr>
<td>Cubic</td>
<td>3</td>
</tr>
</tbody>
</table>

As before, for the sake of clarity, we focus on the univariate sequences. Since \( \phi_n \) is compactly supported, the integral can be efficiently evaluated via an analog anti-aliasing operation. Additionally, without loss of generality, we assume that \( \phi_n \) is orthogonal since there is no need to perform the digital processing step.

The overall rendering process can be efficiently evaluated via an analog anti-aliasing operation. In this section, we assume that the PSF can be designed to counter the effect of the non-ideal acquisition.

Figure 2: The rendering pipeline for minimum-error image representation. For each pixel, the acquisition step numerically evaluates the inner product integral by tracing several rays through the support of the generator. The resulting sequence is then digitally filtered to yield the final pixel values. When \( \phi \) is shift-invariant, Traces several rays through the support of the shifted generator. AKA antialiasing, smooths out high frequencies.
Rendering

\[ f_{\text{app}}(x) = \sum_{n}(r * a^{-1})[n] \varphi(x - n) \]

- Convolve with the inverse autocorrelation sequence.
- Restores some high frequencies.
- Efficiently implemented in the Fourier domain via the FFT.

Continuous Image  \rightarrow  Acquisition  \rightarrow  Digital Image

- Traces several rays through the support of the shifted generator.
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Rendering

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Continuous Image → Acquisition → Digital Image

- Traces several rays through the support of the shifted generator.
- AKA antialiasing, smooths out high frequencies.
- In most cases, simply sample \( f_{app} \) at the integers to obtain the final image.

**Table 1:** Example spaces used in our experiments

<table>
<thead>
<tr>
<th>Space</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic</td>
<td>Compact generator, smooths out high</td>
</tr>
<tr>
<td>Tent</td>
<td>frequencies, efficiently implemented in</td>
</tr>
<tr>
<td>Box</td>
<td>the Fourier domain via the FFT.</td>
</tr>
</tbody>
</table>
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Test Scene

**Experimental setup:**
- Low discrepancy sampler (256 rays per coefficient)
- Rendered at a resolution of 1000 x 1000
- Low dynamic range

<table>
<thead>
<tr>
<th>Existing</th>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>conv. tent</td>
<td>min. error tent</td>
</tr>
<tr>
<td>MN</td>
<td>min. error cubic</td>
</tr>
</tbody>
</table>
Linear Generators

- Effective anti-aliasing
- Smooths out detail in other areas

- Restores high frequency detail
- Introduces Moiré patterns
Linear Generators

• Effective anti-aliasing
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results

conv. tent

min. error tent

difference image
Cubic Generators

MN

min. error cubic

- Effective anti-aliasing
- Slightly sharper than conv. tent

- Reduces Moiré patterns
- Slightly blurrier than min. error tent
Cubic Generators

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difference image
Removing Moiré Patterns

Since $f_{app}(x)$ is continuous, it can be sampled at off-center points to diminish Moiré patterns.

- Effective antialiasing without unduly sacrificing sharpness.
Removing Moiré Patterns

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- Effective antialiasing without unduly sacrificing sharpness.

### Table 2: Percentage of total energy

<table>
<thead>
<tr>
<th>Filter</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box</td>
<td>16.56</td>
</tr>
<tr>
<td>Conv. tent</td>
<td>15.19</td>
</tr>
<tr>
<td>Min-error tent</td>
<td>19.18</td>
</tr>
<tr>
<td>Mitchell-Netravali</td>
<td>15.50</td>
</tr>
<tr>
<td>Min-error cubic</td>
<td>17.41</td>
</tr>
<tr>
<td>Min-error tent (AA)</td>
<td>15.97</td>
</tr>
<tr>
<td>Min-error cubic (AA)</td>
<td>16.34</td>
</tr>
</tbody>
</table>

Energy percentage (measured via the FFT) that lies in the high-pass regime.
Not-so-synthetic Scenes

- Scenes that do not contain pathologically high frequencies have a clear advantage.
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Limitations

- Currently limited to LDR images. Further analysis is needed to extend to HDR images.

- Min. error approximation recovers high frequencies. Therefore, a relatively noise-free acquisition step is necessary.

Min. error approximation leads to overshoots and undershoots across discontinuities.
Future Work

- Extend to hexagonal grids, dynamic scenes
- Multiresolution
- Exploit sparsity, compressive sensing
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Thank you for your attention

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