Compactly Supported Biorthogonal Wavelet Bases on the BCC Lattice

Joshua Horacsek, Usman Alim
Joshua.horacsek@ucalgary.ca
June 12th, 2017
Motivation
- What non-Cartesian computing is, and why should you care about it

Background
- Box splines and biorthogonal wavelet filter banks

Methodology
- Let’s make a wavelet

Results
- Let’s compress data
Motivation
Planar Cartesian Lattice

Hexagonal Lattice
Why bother with non-Cartesian schemes?

- Indexing of data is ambiguous
- Interpolation is more difficult
- Data processing is more difficult

Benefits

- Better approximation properties
- Higher level of isotropy
- Fewer samples needed for reconstructions
Motivation

Cubic Cartesian Lattice

Body Centered Cubic (BCC) Lattice
What has been done?

- Data reconstruction [Xu et al. 2012]
- Gradient approximation [Alim et al. 2010, Hossain et al. 2011]
- Fast Fourier Transforms [Alim et al. 2009]

Wavelets are still missing

- Compression
- De-noising
- Progressive Rendering
Background

— Box splines: natural interpolants on non-Cartesian grids
— Biorthogonal wavelet filter banks: decompose high and low frequencies
Box Splines

- Multivariate extension to B-splines
- Compact, Smooth
- Piecewise Polynomial

To define a box spline, start with a direction matrix

$$\Xi = \begin{bmatrix} \vec{\xi}_1 & \vec{\xi}_2 & \ldots & \vec{\xi}_n \end{bmatrix}$$

$\vec{\xi}_i$ is a two or three dimensional column vector in our case
Box Splines

Convolutional definition, similar to B-splines

\[ M_\Xi(x) = \int_0^1 M_{\Xi \setminus \xi}(x - t\xi) \, dt \quad \text{with} \quad M_{[]} (x) = \delta(x) \]

Some box splines are very natural interpolants for certain lattices

- Box function
  \[
  \begin{bmatrix}
  1 & 0 \\
  0 & 1 \\
  \end{bmatrix}
  \]
- Courant element
  \[
  \begin{bmatrix}
  1 & 0 & 1 \\
  0 & 1 & 1 \\
  \end{bmatrix}
  \]
- Zwart Powell element
  \[
  \begin{bmatrix}
  1 & 0 & 1 & -1 \\
  0 & 1 & 1 & 1 \\
  \end{bmatrix}
  \]
Biorthogonal Wavelet Filter Banks

— A motivating example, in one dimension

- Original function (in Fourier domain)
- After one level of decomposition
- After two levels of decomposition
Wavelet Filter Banks

- In 3D, for a uniform scaling factor of two, we need 8 channels, 1 low pass, 7 high pass
- Biorthogonal Wavelet Filter Banks
  - Decompose a signal in terms of high and low frequency content (enforce perfect reconstruction)
  - Preferably compact filters
  - Number of wavelets depends on number of channels

\[ \sum_j M_j(z^{-1}e^{-\text{i} \pi i}) \tilde{M}_j(z) = \delta[i] \quad \text{for} \quad i \in \{0, \ldots, 7\} \]
Our Methodology

- There are a lot of degrees of freedom
- We can exploit some of the symmetry of the BCC lattice
- We force 7 filters to be geometrically similar, then complete the system of filters, similar to [Cohen et al. 1993]
Wavelet Filter Banks on the BCC Lattice
Wavelet Filter Banks on the BCC Lattice

— Proposition 3.1 tells us that we can obtain the primal low pass, from these dual high pass filters

It’s a box spline!
- Wavelet Filter Banks
Wavelet Filter Banks on the BCC Lattice

— Once we have this primal filter, we need a dual filter
— Proposition 3.1 gives the condition that such a filter must satisfy
— We then solve $A(z)b(z) = (1,0,0,0,0,0,0)\,^T$, which gives the entire filter family
— To get higher order filters, we can also use the bootstrapping procedure of [Cohen et al. 1993]
- Wavelet Filter Banks
Higher order filters

- Some applications need higher order filters (filters that are more smooth in the limit)
- Boot-strapping (juggle filters and convolutions)
Experiments

- Compression via Thresholding
  - Perform hierarchical wavelet decomposition
  - Discard coefficient such that only a percentage of the largest coefficients remain
  - Reconstruct the volumetric image
Experiments

Datasets

- Start with high resolution data sets, resample them onto a lower resolution Cartesian and BCC grid
- Sampling density is roughly the same
- Marschner Lobb is known in closed form
- Measure peak signal to noise ratio (PSNR)
Qualitative Results

— Head Dataset

BCC

Original
Sub-sampled

75% Discarded

85% Discarded

95% Discarded

CC
- Qualitative Results (95% Removed)

Trilinear Cartesian
Linear BCC
Higher order BCC
Qualitative Results

- Marschner Lobb function
Experiments

- Qualitative Results (95% Removed)

- Trilinear Cartesian
- Linear BCC
- Higher order BCC
Quantitative Results

Head Dataset

Approx PSNR vs % Details

- CC-CDF 2,2
- BCC 1,1
- BCC 2,1

Experiments
Quantitative Results

Marschner Lobb Function

Approx PSNR vs % Details

- CC–CDF 2,2
- BCC 1,1
- BCC 2,1
Contributions

— We derived a family of geometrically appropriate wavelet filter banks on the BCC lattice
— Our wavelet filter banks perform at least on par with Cartesian style CDF wavelets
References